First Midsemestral Examination
B.Math. Hons. Ist year
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Be brief !

Q 1.
Define the limit superior ($\lim Sup a_n$) of a bounded sequence $\{a_n\}$. Prove that if $l = \lim$ Sup a_n, then for each $\epsilon > 0$, there exists a subsequence $\{a_{n_k}\}_k$ such that $a_{n_k} > l - \epsilon$ for all k.

OR

If $\{a_n\}$ is a sequence for which every convergent subsequence converges to the same limit l, prove that $\{a_n\} \to l$ as $n \to \infty$.

Q 2.
Define what is meant by a compact subset A of $\mathbf{R}$. Prove that a compact subset A must be closed.

Q 3.
Prove that $s =$Sup $\{x \in \mathbf{R} : x^2 < 3\}$ exists and that $s^2 = 3$.

OR

Let $\{a_n\}$ be a bounded sequence and let $s =$Sup $\{a_n\}$. If $s \neq a_n$ for any n, then show that there exists a subsequence of $\{a_n\}$ which converges to s.

Q 4.
Define a Cauchy sequence and prove that such a sequence must be bounded.

Q 5.
Find (with proof) $\lim_{n\to\infty}(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2})$.

OR

Prove $\lim_{n\to\infty} \frac{n!}{n^n}$ exists and equals 0.
Hint : $a_n := \frac{n!}{n^n}$ is $\downarrow$ and look at $\frac{a_{n+1}}{a_n}$.

Q 6.
Prove that $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$ converges.